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A SYSTEM OF AXIOMATIC SET THEORY—PART I

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Introduction. The system of axioms for set theory to be exhibited in this paper[1] is a modification of the axiom system due to von Neumann.[2] In particular it adopts the principal idea of von Neumann, that the elimination of the undefined notion of a *property* ("definite Eigenschaft"),[3] which occurs in the original axiom system of Zermelo,[4] can be accomplished in such a way as to make the resulting axiom system elementary, in the sense of being formalizable in the logical calculus of first order, which contains no other bound variables than individual variables and no accessory rule of inference (as, for instance, a scheme of complete induction).[5]

The purpose of modifying the von Neumann system is to remain nearer to the structure of the original Zermelo system and to utilize at the same time some of the set-theoretic concepts of the Schröder logic and of *Principia mathematica* which have become familiar to logicians. As will be seen, a considerable simplification results from this arrangement.

The theory is not set up as a pure formalism, but rather in the usual manner of elementary axiom theory, where we have to deal with propositions which are understood to have a meaning, and where the reference to the domain of facts to be axiomatized is suggested by the names for the kinds of individuals and for the fundamental predicates.

On the other hand, from the formulation of the axioms and the methods used in making inferences from them, it will be obvious that the theory can be formalized by means of the logical calculus of first order ("Prädikatenkalkul" or

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[1] This system was first introduced by the author in a lecture on "Mathematical Logic" at the University of Göttingen, 1929-30.

[2] J. v. Neumann, *Eine Axiomatisierung der Mengenlehre*, ***Journal für die reine und angewandte Mathematik***, vol. 154 (1925), pp. 219–240; *Die Axiomatisierung der Mengenlehre*, ***Mathematische Zeitschrift***, vol. 27 (1928), pp. 669–752; *Über eine Widerspruchsfreiheitsfrage in der axiomatischen Mengenlehre*, ***Journal r. angew. Math.***, vol. 160 (1929), pp. 227–241.

[3] This elimination was first carried out, in two different ways, by Th. Skolem and A. Fraenkel. See Th. Skolem, *Einige Bemerkungen zur axiomatischen Begründung der Mengenlehre*, ***Wissenschaftliche Vorträge gehalten auf dem 5. Kongress der skandinavischen Mathematiker in Helsingfors 1922***, Helsingfors 1923, pp. 217–232; and A. Fraenkel, *Zu den Grundlagen der Cantor-Zermeloschen Mengenlehre*, ***Mathematische Annalen***, vol. 86 (1922), pp. 230–237; *Untersuchungen über die Grundlagen der Mengenlehre*, ***Mathematische Zeitschrift***, vol. 22 (1925), pp. 250–273; ***Zehn Vorlesungen über die Grundlegung der Mengenlehre***, Leipzig and Berlin 1927; ***Einleitung in die Mengenlehre***, 3rd. edn., Berlin 1928.

[4] E. Zermelo, *Untersuchungen über die Grundlagen der Mengenlehre I*, ***Mathematische Annalen***, vol. 65 (1908), pp. 261–281.

[5] It may be observed that the von Neumann axiom system for set theory is the first example of an axiom system which is at once adequate to arithmetic and elementary in the sense just described.

"engere Funktionenkalkül") with the addition of the formalism of equality and the ι-symbol[6] for "descriptions" (in the sense of Whitehead and Russell).

1. Fundamental concepts, notations, derived notions. According to the leading idea of the von Neumann set theory we have to deal with two kinds of individuals, which we may distinguish as *sets* and *classes*. The distinction may be thought of in this way, that a set is a multitude forming a proper thing, whereas a class is a predicate regarded only with respect to its extension.

We shall indicate this distinction by using small italics to refer to sets and capital italics for classes (the letters $a, \cdots, t, A, \cdots, T$ will be used as free variables, $u, \cdots, z, U, \cdots, Z$ as bound variables). Sometimes a letter will be chosen as a proper name (symbol) for a particular set or class, and in this case a small Greek letter will be taken for a set, a capital Greek letter for a class.

To denote expressions (including, in particular, expressions consisting of a single letter), German letters will be used.

"Being the same individual" will be denoted by the symbol $=$. As in the usual postulate theory, the identity, $=$, is not regarded as a primitive predicate of the system, but as a logical concept immediately connected with the idea of an individual.

Corresponding to the two kinds of individuals we have two primitive relations, one between sets,

$$a \epsilon b,$$

to be read *a is in b* or *a is an element of b*, and one between a set and a class,

$$a \eta B,$$

to be read, *a belongs to B* or *a is an element of B*. We assume that for any sets a, b it is uniquely determined whether $a \epsilon b$ or not, and that for a set a and a class B it is always uniquely determined whether $a \eta B$ or not.

Observe that a class never occurs as an element; nor are there elements of any other kind than sets.[7]

The relations ϵ, η are the only primitive predicates of our system. Some derived notions are immediately obtained from them by applying logical terms. Thus $a \mathrel{-\!\!\in} b$, or *a is a subset of b*, means that every element of a is also an element of b; and $A \mathrel{-\!\!\in} B$, or *A is a subclass of B*, means that every element of A is also an element of B; and the relations $a \mathrel{-\!\!\in} B$ and $A \mathrel{-\!\!\in} b$ are to be defined in quite the same way. Likewise, $a \subset b$, or *a is a proper subset of b*, means that $a \mathrel{-\!\!\in} b$ but not $b \mathrel{-\!\!\in} a$; and $A \subset B$, or *A is a proper subclass of B*, means that $A \mathrel{-\!\!\in} B$ but not

[6] Cf. Hilbert and Bernays, ***Grundlagen der Mathematik I***, 1934, §8.

[7] The original Zermelo system admits the existence of elements which are not sets. Zermelo insists on this point for the sake of generality. And in his recent axiomatization of set theory (*Über Grenzzahlen und Mengenbereiche*, ***Fundamenta mathematicae***, vol. 16 (1930), pp. 29–47) he explicitly introduces *Urelemente*.

In the systems of Fraenkel and v. Neumann, on the other hand, it is assumed that every element is a set. This idea of avoiding elements which are not sets was apparently first suggested by P. Finsler.

Whether the one procedure or the other is preferable depends on the purpose for which the system is intended.

$B \mathrel{-\!\!\in} A$; and the relations $a \subset B$ and $A \subset b$ are to be defined in the same way.

That the set a *represents* the class A means that $a \mathrel{-\!\!\in} A$ and $A \mathrel{-\!\!\in} a$. A set c is said to be *reflexive* if $c \epsilon c$. A set c is said to be *transitive* if every element of an element of c is also an element of c. A set or class is said to be *empty* if there is no element of it.

In order to make clear the logical character of these definitions, it is desirable to write them down by means of the logical symbols:

$\overline{\mathfrak{A}}$	"Not $\mathfrak{A}$"	(negation)
$\mathfrak{A}$ & $\mathfrak{B}$	"$\mathfrak{A}$ and $\mathfrak{B}$"	(conjunction)
$\mathfrak{A} \vee \mathfrak{B}$	"$\mathfrak{A}$ or $\mathfrak{B}$"	(disjunction)
$\mathfrak{A} \rightarrow \mathfrak{B}$	"If $\mathfrak{A}$, then $\mathfrak{B}$"	(implication)
$(\mathfrak{w})\mathfrak{A}(\mathfrak{w})$	"For every set $\mathfrak{w}$, $\mathfrak{A}(\mathfrak{w})$"	(universal quantification)
$(E\mathfrak{w})\mathfrak{A}(\mathfrak{w})$	"For some set $\mathfrak{w}$, $\mathfrak{A}(\mathfrak{w})$"	(existential quantification)

Definition of

$$a \mathrel{-\!\!\in} b: \quad (x)(x \epsilon a \rightarrow x \epsilon b).$$
$$a \mathrel{-\!\!\in} B: \quad (x)(x \epsilon a \rightarrow x \eta B).$$
$$A \mathrel{-\!\!\in} b: \quad (x)(x \eta A \rightarrow x \epsilon b).$$
$$A \mathrel{-\!\!\in} B: \quad (x)(x \eta A \rightarrow x \eta B).$$
$$a \subset b: \quad a \mathrel{-\!\!\in} b \;\&\; \overline{b \mathrel{-\!\!\in} a}.$$

Similarly $a \subset B$, $A \subset b$, $A \subset B$ are to be defined.

a represents A:	$a \mathrel{-\!\!\in} A$ & $A \mathrel{-\!\!\in} a$.
c is reflexive:	$c \epsilon c$.
c is transitive:	$(x)(y)(x \epsilon y \;\&\; y \epsilon c \rightarrow x \epsilon c)$.
c is empty:	$\overline{(Ex)}(x \epsilon c)$, or $(x)\overline{(x \epsilon c)}$.
c is non-empty:	$(Ex)(x \epsilon c)$.

Similarly C *is empty* is to be defined.

In regard to the formalization of the identity, it may be remarked that, for deductive operation with the symbol $=$, the following formulas, to be used as initial formulas (formal axioms) are sufficient:

$$a = a.$$
$$a = b \rightarrow (a \epsilon c \rightarrow b \epsilon c).$$
$$a = b \rightarrow (c \epsilon a \rightarrow c \epsilon b).$$
$$a = b \rightarrow (a \eta C \rightarrow b \eta C).$$

Instead of $\overline{a = b}$ we shall usually write $a \neq b$ ("a is another set than b").

2. The axioms, first part. We shall now state the axioms of the system which we are presenting. As in the Hilbert system of axiomatic geometry, the axioms are distributed in several series.

The first axioms are almost the same as in the Zermelo system.

I. Axioms of extensionality.

(1) If the set a has the same elements as the set b, then a is the same set as b. (This can be expressed by the formula $a \mathrel{-\!\!\in} b \;\&\; b \mathrel{-\!\!\in} a \rightarrow a = b$.)

(2) If the class A has the same elements as the class B, then A is the same class as B.

Remarks.

1. Instead of employing the logical concept of identity and introducing the axiom I(1), there would be, as A. Fraenkel pointed out,[8] the possibility of introducing equality as a derived notion, by defining $a=b$ as $a \mathrel{-\!\!\in} b$ & $b \mathrel{-\!\!\in} a$, and taking as axioms the properties of equality which are expressed by the formulas,

$$a=b \rightarrow (a \epsilon c \rightarrow b \epsilon c),$$
$$a=b \rightarrow (a \eta C \rightarrow b \eta C).$$

This reduction, or a similar one, may be useful in the investigation of consistency. But for setting up the theory the method of defining the equality $a=b$ does not seem to be advantageous, since the meaning of several of the axioms is complicated by it.

2. The axiom I(2) will be used only for the purpose of defining a class by saying what elements belong to it. Since such a definition is a case of a description and, according to a theorem of logic,[9] descriptions generally can be eliminated, the introduction of the axiom I(2) could be avoided, and we could get along without speaking anywhere of identity between classes.

At all events, if the system of our axioms with exception of I(2) can be shown to be consistent, the consistency including this axiom is a consequence.

II. Axioms of direct construction of sets.

(1) There exists a set which has no element.

(2) To a set s can be adjoined as element any set c which is not already in s. In other words, given a set s and a set c not in s, there exists a set t, such that $s \subset t$ and c is the only element of t which is not in s.

Immediate consequences, notations. Combining the axioms II(1), (2) with I(1), we find that for both the former axioms the set asserted to exist can be shown also to be uniquely determined. Thus we have:

1. There is a unique set characterized by the property of having no element. It may be called the *null set* and denoted by 0.

2. For any two sets s and c there is a unique set characterized by the property that its elements are those and only those sets which either are in s or are identical with c. (Observe that the case $c \epsilon s$ need not be excluded, since in this case s itself is the unique set having the required property.)

In particular, taking $s=0$, we find that, corresponding to any set c, there is a unique set whose only element is c. It may be denoted by (c).

Similarly, taking $s=(a)$, we find that, corresponding to any two sets a and b, there is a unique set whose only elements are a and b. This set will be denoted by (a, b). (If $a=b$, then $(a, b)=(a)$.)

The set $((a), (a, b))$, which is uniquely determined by the sets a, b (in the given order), will be called the *pair* a, b and denoted by $\langle a, b \rangle$.[10] Observe that the

[8] A. Fraenkel, *Über die Gleichheitsbeziehung in der Mengenlehre*, **Journal für die reine und angewandte Mathematik,** vol. 157 (1927), pp. 79–81.

[9] Cf. Hilbert and Bernays, **Grundlagen der Mathematik I,** §8.

[10] This manner of representing the ordered pair is due to C. Kuratowski (*Sur la notion de l'ordre dans la théorie des ensembles*, **Fundamenta mathematicae,** vol. 2 (1921), pp. 161–171).

relation, $\langle a, b\rangle = c$, as well as the assertion, "*c is a pair*," can be expressed by means of ϵ, $=$, and our logical symbols.

If $\langle a, b\rangle = \langle c, d\rangle$, then $a=c$ and $b=d$. Indeed, for $c=d$ we have:

$$\begin{aligned} &\langle c, d\rangle = ((c), (c, d)) = ((c));\\ \therefore\quad &((a), (a, b)) = (c),\\ &(a) = (a, b) = (c),\\ &a=c, \quad b=c, \quad b=d. \end{aligned}$$

And for $c \neq d$ we have:

$$\begin{aligned} &(c) \neq (c, d), \quad (a) \neq (c, d);\\ &(a) \epsilon ((c), (c, d)), \quad (a) = (c), \quad a=c;\\ &(c, d) \epsilon ((a), (a, b)), \quad (a, b) = (c, d),\\ &(a) \neq (a, b), \quad a \neq b, \quad b \neq c, \quad b=d. \end{aligned}$$

Thus if $\langle a, b\rangle = c$, the sets a, b are uniquely determined by the set c. We call a the *first member*, and b the *second member*, of the pair $\langle a, b\rangle$.

The pair $\langle b, a\rangle$ will be said to be *converse* to $\langle a, b\rangle$. The pair $\langle\langle a, b\rangle, c\rangle$ will be said to arise from $\langle a, \langle b, c\rangle\rangle$ by *coupling to the left*, and $\langle a, \langle b, c\rangle\rangle$ will be said to arise from $\langle\langle a, b\rangle, c\rangle$ by *coupling to the right*.

III. Axioms for construction of classes.

a(1) Corresponding to any set a there exists a class whose only element is a.

a(2) Corresponding to any class A there exists a class to which a set belongs if and only if it does not belong to A (*complementary class to* A).

a(3) Corresponding to any two classes A, B there exists a class to which a set belongs if and only if it belongs both to A and to B (*intersection* of A and B, or, as we shall also say, of A with B).

b(1) There exists a class whose elements are those sets which have one and only one element.

b(2) There exists a class whose elements are those pairs $\langle a, b\rangle$ for which $a \epsilon b$.

b(3) Corresponding to any class A there exists a class whose elements are those pairs $\langle a, b\rangle$ for which $a \eta A$.

c(1) Corresponding to any class A of pairs there exists a class whose elements are the first members of the elements of A (*domain* of A).

c(2) Corresponding to any class A of pairs there exists a class whose elements are the pairs which are converse to the elements of A (*converse class* to A).

c(3) Corresponding to any class A of pairs of the form $\langle a, \langle b, c\rangle\rangle$ there exists a class whose elements are the pairs which arise from the elements of A by coupling to the left.

Immediate consequences. In consequence of I(2), each of the assertions of the axioms III can be supplemented by the remark that there is only one class having the postulated property. Hence we may speak of *the* class whose only element is a, *the* complementary class to A, *the* intersection of A and B, and so on.

Combining the axioms III among themselves, we are led to further classes, which again are uniquely determined in consequence of I(2):

1. The intersection of the class whose only element is 0 with the class whose only element is (0) is the *empty class*. The complementary class to this is the *class of all sets*. The class of the pairs $\langle a, b\rangle$ for which a belongs to the class of all sets is the *class of all pairs*.

2. The complementary class of the intersection of the complementary classes of A and B has as its elements the sets a characterized by the property, "Either a belongs to A or a belongs to B." We call it the *sum* of A and B.

3. The converse class to the class of pairs whose first member belongs to B is the class of pairs whose second member belongs to B. The intersection of the class of pairs whose first member belongs to A and the class of pairs whose second member belongs to B is the class of pairs $\langle a, b\rangle$ such that $a\eta A$ and $b\eta B$.

4. The domain of the converse class of a class A of pairs is the class of second members of elements of A. We call it the *converse domain* of A. The sum of the domain and the converse domain of a class A of pairs is the *field* of A. Its elements are the members of the elements of A.

5. Starting from a pair $\langle\langle a, b\rangle, c\rangle$, passing to the converse, then coupling to the left, then passing again to the converse, coupling again to the left, and finally again passing to the converse, we get the pair $\langle a, \langle b, c\rangle\rangle$. Thus from a class A of pairs of the form $\langle\langle a, b\rangle, c\rangle$, by application of III c(2) and c(3) we obtain the class of pairs which arise from elements of A by coupling to the right—i.e. the process of applying III c(3) can be inverted. The process of applying III c(3) may be denoted briefly as *coupling to the left*, the inverse process as *coupling to the right*.

6. By III a(3), b(1), b(2), b(3), c(2) we may obtain the intersection of the class of pairs $\langle a, b\rangle$ such that $a\epsilon b$ and the class of pairs $\langle a, b\rangle$ such that a and b have each one and only one element. The converse domain of this intersection has as its elements the sets of the form $((c))$. But since $((c))$ is the same as $\langle c, c\rangle$, this gives us the class of all pairs $\langle a, b\rangle$ such that $a=b$.[11]

7. Let c be a set. By III a(1), a(3), b(2), b(3), c(2) we may obtain the intersection of the class of pairs $\langle a, b\rangle$ such that $a\epsilon b$ and the class of pairs whose second member is c. The domain of this intersection has the same elements that c has. Thus every set represents a class. But, as we shall see, not every class is represented by a set.

3. Predicates and classes. Before going on to consideration of the remaining axioms it will be desirable to have a certain survey of the consequences of the axioms III. For this purpose we shall prove a metamathematical theorem concerning the possibility of making classes correspond to certain predicates of sets, the term *predicate* being taken in the wider sense, so as to include predicates of several subjects (relations).

The predicates here in question are defined by means of certain expressions, which, in general, contain besides the *arguments*, or variables corresponding to subjects, still other variables as *parameters*.

These expressions are the following: first the *primary expressions*, $\mathfrak{a}\epsilon\mathfrak{b}$, and $\mathfrak{a}=\mathfrak{b}$, and $\mathfrak{a}\eta\mathfrak{B}$, where $\mathfrak{a}$ and $\mathfrak{b}$ denote free variables for sets (small italics) and $\mathfrak{B}$ denotes a free variable for a class (capital italics); further the expressions obtainable from primary expressions by the logical operations, conjunction, dis-

[11] This inference depends on the special form of the definition which we have adopted for the ordered pair. We could avoid this dependency by taking instead of our axiom III b(1) an axiom saying that there exists a class whose elements are the pairs of the form $\langle c, c\rangle$.

junction, implication, negation, and the quantifiers, every quantifier changing a free variable into a bound one, but with the restriction that the quantifiers are to be applied only to variables for sets.

These expressions will be called here *constitutive expressions*.

A constitutive expression, with some of the free variables *for sets* taken as arguments and all other free variables occurring in it taken as parameters, represents, for each system of fixed values of the parameters, a predicate with the places of the subjects marked by the arguments. For instance the expression.

$$(Ex)(a \epsilon x \ \& \ x \epsilon b \ \& \ x \eta C),$$

if b and C are taken as parameters, represents the following property of a set a: to be in a set which is a common element of the set b and the class C.

Now it is a question of making classes correspond to the predicates defined by constitutive expressions. For this purpose we need the notion of a *k-tuplet* ($k=1, 2, 3, \cdots$).

A *k-tuplet* is a set, formed ouf of k sets $\mathfrak{a}_1, \cdots, \mathfrak{a}_k$, the *members* of the k-tuplet, by the iterated operation of forming pairs, as follows. A 1-tuplet (singlet) formed out of $\mathfrak{a}$ is $\mathfrak{a}$ itself. A $(k+1)$-tuplet formed out of $\mathfrak{a}_1, \cdots, \mathfrak{a}_{k+1}$ is a pair $\langle \mathfrak{u}, \mathfrak{v} \rangle$, where $\mathfrak{u}$ is a p-tuplet, $\mathfrak{v}$ is a q-tuplet, $p+q=k+1$, the members of $\mathfrak{u}$ are some p of the sets $\mathfrak{a}_1, \cdots, \mathfrak{a}_{k+1}$, and the members of $\mathfrak{v}$ are the remainder of those sets.

Under this definition a 2-tuplet (doublet) formed out of $\mathfrak{a}$ and $\mathfrak{b}$ is one or other of the pairs $\langle \mathfrak{a}, \mathfrak{b} \rangle$, $\langle \mathfrak{b}, \mathfrak{a} \rangle$ and the members of the doublet are the same as the members of the pair under our previous definition.

Observe that a set c which is a $(k+1)$-tuplet is also a k-tuplet, but that the members of c as a k-tuplet are not the same as the members of c as a $(k+1)$-tuplet.

If in the formation of a k-tuplet we replace the members in order by the variables $a_1, \cdots, a_k$, we obtain the *schema* of the k-tuplet. Thus the schema of a quadruplet $\langle \langle \mathfrak{a}, \mathfrak{b} \rangle, \langle \mathfrak{c}, \mathfrak{d} \rangle \rangle$ is $\langle \langle a_1, a_2 \rangle, \langle a_3, a_4 \rangle \rangle$.

The number of different k-tuplet schemata formed out of the variables $a_1, \cdots, a_k$ can be shown to be

$$\frac{(2k-2)!}{(k-1)!\,k!}.$$

In the schema of a k-tuplet each of the variables has a *degree*, i.e. the number of brackets $\langle \ \rangle$ by which it is enclosed. In the k-tuplet itself we define the degree of a member to be the degree of the corresponding variable in the schema of the k-tuplet.

A k-tuplet will be called normal if, in the succession of members $\mathfrak{a}_1, \cdots, \mathfrak{a}_k$, each member $\mathfrak{a}_p$, where $p<k$, has the degree p, and $\mathfrak{a}_k$ has the degree $k-1$. Thus, for instance, a normal quintuplet has the form $\langle \mathfrak{a}_1, \langle \mathfrak{a}_2, \langle \mathfrak{a}_3, \langle \mathfrak{a}_4, \mathfrak{a}_5 \rangle \rangle \rangle \rangle$. (All singlets and doublets are normal.)

The possible differences between k-tuplets formed out of the same members consist in the order of the members and the positions of the brackets $\langle \ \rangle$. A change in the positions of the brackets, preserving the order of the members, will be called a *rebracketing*.

A k-tuplet will be said to *correspond* to a series (system of values) $\mathfrak{a}_1, \cdots, \mathfrak{a}_k$ if $\mathfrak{a}_1, \cdots, \mathfrak{a}_k$ are the members of the k-tuplet in order.

Now the theorem to be proved can be formulated as follows: *If* $\mathfrak{P}(\mathfrak{v}_1, \cdots, \mathfrak{v}_k)$ *is a constitutive expression with* $\mathfrak{v}_1, \cdots, \mathfrak{v}_k$ *as arguments, then for every system of fixed values of the parameters there exists, by virtue of the axioms* III, *a class the elements of which are the normal k-tuplets corresponding to those systems of values of* $\mathfrak{v}_1, \cdots, \mathfrak{v}_k$ *for which the predicate* $\mathfrak{P}(\mathfrak{v}_1, \cdots, \mathfrak{v}_k)$ *holds for the fixed values in question of the parameters.*

In order to prove this we consider first the case that $\mathfrak{P}(\mathfrak{v}_1, \cdots, \mathfrak{v}_k)$ is a primary expression. Then it has one of the forms $\mathfrak{v}_1 \eta \mathfrak{C}$, $\mathfrak{v}_1 \epsilon \mathfrak{r}$, $\mathfrak{r} \epsilon \mathfrak{v}_1$, $\mathfrak{v}_1 = \mathfrak{r}$, $\mathfrak{r} = \mathfrak{v}_1$, $\mathfrak{v}_1 \epsilon \mathfrak{v}_2$, $\mathfrak{v}_2 \epsilon \mathfrak{v}_1$, $\mathfrak{v}_1 = \mathfrak{v}_2$, $\mathfrak{v}_2 = \mathfrak{v}_1$, where $\mathfrak{C}$ denotes a parameter referring to classes and $\mathfrak{r}$ a parameter referring to sets.

In all of these cases the existence of a class with the required property is easily seen, since the normal singlet corresponding to a value of $\mathfrak{v}$ is just that value itself, and the normal doublet corresponding to a system of values $\mathfrak{a}_1$, $\mathfrak{a}_2$ of $\mathfrak{v}_1$, $\mathfrak{v}_2$ is the pair $\langle \mathfrak{a}_1, \mathfrak{a}_2 \rangle$. Indeed, for $\mathfrak{v}_1 \eta \mathfrak{C}$ the class taken as value for $\mathfrak{C}$ is the required class. For $\mathfrak{v}_1 = \mathfrak{r}$ or $\mathfrak{r} = \mathfrak{v}_1$ the existence of a class with the required property follows immediately from III a(1). For $\mathfrak{v}_1 \epsilon \mathfrak{v}_2$ or $\mathfrak{v}_2 \epsilon \mathfrak{v}_1$ it follows from III b(2), c(2). For $\mathfrak{v}_1 = \mathfrak{v}_2$ or $\mathfrak{v}_2 = \mathfrak{v}_1$ it follows from the consequence 6 of the axioms III, derived above (§2). For $\mathfrak{v}_1 \epsilon \mathfrak{r}$ it follows from consequence 7 of the axioms III. And in a similar way it follows for $\mathfrak{r} \epsilon \mathfrak{v}_1$ by the axioms III a(1), a(3), b(2), b(3), c(1), c(2).

Now going on to constitutive expressions formed by means of the logical operations, we first remark that on account of the equivalence of $\mathfrak{A} \vee \mathfrak{B}$ to $\overline{\overline{\mathfrak{A}} \,\&\, \overline{\mathfrak{B}}}$, of $\mathfrak{A} \rightarrow \mathfrak{B}$ to $\overline{\mathfrak{A} \,\&\, \overline{\mathfrak{B}}}$, and of $(\mathfrak{w})\mathfrak{A}(\mathfrak{w})$ to $\overline{(E\mathfrak{w})\overline{\mathfrak{A}(\mathfrak{w})}}$ we need consider only the three operations, conjunction, negation, and existential quantification. Concerning these three operations we have at once the following:

1. If C is a class of the required property with respect to a constitutive expression $\mathfrak{P}(\mathfrak{v}_1, \cdots, \mathfrak{v}_k)$ and a fixed system of values for the parameters, then the complementary class of C has the required property with respect to $\overline{\mathfrak{P}(\mathfrak{v}_1, \cdots, \mathfrak{v}_k)}$ and the same system of values for the parameters.

2. If C is a class of the required property with respect to a constitutive expression $\mathfrak{P}(\mathfrak{v}_1, \cdots, \mathfrak{v}_k)$ $(k>1)$ and a fixed system of values for the parameters, then the converse domain of C has the required property with respect to the expression $(E\mathfrak{u})\mathfrak{P}(\mathfrak{u}, \mathfrak{v}_2, \cdots, \mathfrak{v}_k)$ and the same system of values for the parameters, $\mathfrak{u}$ being a variable which does not occur in $\mathfrak{P}(\mathfrak{v}_1, \cdots, \mathfrak{v}_k)$.

3. If $\mathfrak{P}(\mathfrak{u}_1, \cdots, \mathfrak{u}_r)$ and $\mathfrak{Q}(\mathfrak{v}_1, \cdots, \mathfrak{v}_s)$ are constitutive expressions which have no argument in common and if, for a fixed system of values of the parameters which occur, A is a class of the required property with respect to $\mathfrak{P}(\mathfrak{u}_1, \cdots, \mathfrak{u}_r)$ and B is a class of the required property with respect to $\mathfrak{Q}(\mathfrak{v}_1, \cdots, \mathfrak{v}_s)$, then by consequence 3 of the axioms III (§2 above) there exists a class whose elements are those $(r+s)$-tuplets $\langle \mathfrak{a}, \mathfrak{b} \rangle$, $\mathfrak{a}$ being a normal r-tuplet corresponding to a system of values $\mathfrak{a}_1, \cdots, \mathfrak{a}_r$, and $\mathfrak{b}$ a normal s-tuplet corresponding to a system of values $\mathfrak{b}_1, \cdots, \mathfrak{b}_s$, for which

$$\mathfrak{P}(\mathfrak{a}_1, \cdots, \mathfrak{a}_r) \,\&\, \mathfrak{Q}(\mathfrak{b}_1, \cdots, \mathfrak{b}_s)$$

holds for the fixed system of values of the parameters.

Looking at these results 1–3 from the point of view of what has to be proved, we see that 1 and 2, concerning negation and existential quantification, are sufficient. (In connection with 2, note that there is no loss of generality in the assumption that the existential quantifier is applied to the first one of the arguments in the given constitutive expression, because the order $\mathfrak{v}_1, \cdots, \mathfrak{v}_k$ of these arguments may be arbitrarily chosen.) In 3, however, two things are lacking: the case that the expressions $\mathfrak{P}(\mathfrak{u}_1, \cdots, \mathfrak{u}_r)$ and $\mathfrak{Q}(\mathfrak{v}_1, \cdots, \mathfrak{v}_s)$ have common arguments is not included, and the class of $(r+s)$-tuplets which is proved to exist is, in the case $r>1$, not a class of normal $(r+s)$-tuplets. Thus in order to complete the proof of our theorem we have still to remove these two deficiencies.

Concerning the first of them we remark that the case of common arguments of the expressions $\mathfrak{P}(\mathfrak{u}_1, \cdots, \mathfrak{u}_r)$ and $\mathfrak{Q}(\mathfrak{v}_1, \cdots, \mathfrak{v}_s)$ can be treated by first taking all the arguments as different from one another and afterwards in the conjunction

$$\mathfrak{P}(\mathfrak{u}_1, \cdots, \mathfrak{u}_r) \;\&\; \mathfrak{Q}(\mathfrak{v}_1, \cdots, \mathfrak{v}_s)$$

identifying some of the arguments. The identifications can be performed successively, so that at each step only two variables are identified. And by performing permutations on the arguments we can arrange that the arguments identified are, at each step, the first two.

In connection with the other deficiency, concerning the form of the $(r+s)$-tuplets in 3, note that these $(r+s)$-tuplets all have the same schema and that each of them corresponds to a system of values of the arguments $\mathfrak{u}_1, \cdots, \mathfrak{u}_r, \mathfrak{v}_1, \cdots, \mathfrak{v}_s$ for which the predicate represented by the expression

$$\mathfrak{P}(\mathfrak{u}_1, \cdots, \mathfrak{u}_r) \;\&\; \mathfrak{Q}(\mathfrak{v}_1, \cdots, \mathfrak{v}_s)$$

holds (for the fixed system of values of the parameters). Thus the transition from the class of $(r+s)$-tuplets in 3 to the class which is to be proved to exist in the case of the constitutive expression

$$\mathfrak{P}(\mathfrak{u}_1, \cdots, \mathfrak{u}_r) \;\&\; \mathfrak{Q}(\mathfrak{v}_1, \cdots, \mathfrak{v}_s)$$

(with $\mathfrak{u}_1, \cdots, \mathfrak{u}_r, \mathfrak{v}_1, \cdots, \mathfrak{v}_s$ all different) will consist only in a rebracketing, the same for each $(r+s)$-tuplet.

Hence for the completion of our proof it will be sufficient to establish the following two things:

4. If in the case of the constitutive expression $\mathfrak{A}(\mathfrak{v}_1, \cdots, \mathfrak{v}_k)$ $(k>1)$ a class with the property required by our theorem exists for a fixed system of values of the parameters, then a class with the required property exists in the case of each of the expressions resulting from $\mathfrak{A}(\mathfrak{v}_1, \cdots, \mathfrak{v}_k)$ by a permutation of the arguments $\mathfrak{v}_1, \cdots, \mathfrak{v}_k$, and also in the case of the expression $\mathfrak{A}(\mathfrak{v}_2, \mathfrak{v}_2, \cdots, \mathfrak{v}_k)$ with the $k-1$ arguments $\mathfrak{v}_2, \cdots, \mathfrak{v}_k$ (where it is understood each time that the values of the parameters are preserved).

5. Corresponding to any class C of k-tuplets which all have the same schema there exists a class the elements of which are the normal k-tuplets obtained from the elements of C by rebracketing.

Moreover, proof of 4 reduces to proof of the two following things:

4a. If the same permutation is applied to the members of each element of a class C of normal k-tuplets, the resulting class of normal k-tuplets also exists.

4b. Corresponding to any class C of normal $(k+1)$-tuplets there exists the class obtained from C by omitting all the $(k+1)$-tuplets in which the first two members are different and then canceling the first member of each of the remaining $(k+1)$-tuplets (so as to obtain a normal k-tuplet).

Of these, 4b may be proved as follows. By the axiom III b(3) and the consequence 6 of the axioms III (see §2 above), there exists the class of all sets of the form $\langle\langle a, a\rangle, b\rangle$, from which we obtain by coupling to the right (consequence 5 of the axioms III) the class of all sets of the form $\langle a, \langle a, b\rangle\rangle$. If C is a class of normal $(k+1)$-tuplets, then in the case $k>1$ the intersection of C with the class of sets having the form $\langle a, \langle a, b\rangle\rangle$ is the class of those $(k+1)$-tuplets of C in which the first two members are equal. In the case $k=1$ the corresponding subclass of C is obtained as the intersection of C with the class of pairs $\langle c, c\rangle$ (consequence 6). Thus in both cases we have a class C^* arising from C by omitting the $(k+1)$-tuplets in which the first member is different from the second. And the converse domain of C^* is the class obtained from C^* by canceling the first member of each $(k+1)$-tuplet.

The assertions 4a and 5 can be combined into the following: If the same permutation is applied to the members of each element of a class C of k-tuplets which all have the same schema and at the same time each element of C is rebracketed so as to render it a normal k-tuplet, the resulting class of normal k-tuplets exists. And in order to prove this it will be sufficient to prove the two following things:

6. The passage, by permutation and rebracketing, from a given k-tuplet to a prescribed normal k-tuplet with the same members, can be performed by a succession of steps of the following kind:

$s_1{}^k$. Replacing a k-tuplet, regarded as a pair, by the converse pair.

$s_2{}^k$. Coupling to the left or to the right, applied to a k-tuplet, i.e., replacing a k-tuplet $\langle\mathfrak{a}, \langle\mathfrak{b}, \mathfrak{c}\rangle\rangle$ by $\langle\langle\mathfrak{a}, \mathfrak{b}\rangle, \mathfrak{c}\rangle$ or inversely.

$s_3{}^k$. Replacing a k-tuplet $\langle\mathfrak{a}, \langle\mathfrak{b}, \mathfrak{c}\rangle\rangle$ by $\langle\mathfrak{a}, \langle\mathfrak{c}, \mathfrak{b}\rangle\rangle$, or a k-tuplet $\langle\langle\mathfrak{a}, \mathfrak{b}\rangle, \mathfrak{c}\rangle$ by $\langle\langle\mathfrak{b}, \mathfrak{a}\rangle, \mathfrak{c}\rangle$, i.e., replacing a pair which is a member of a pair $\mathfrak{p}$ ($\mathfrak{p}$ being a k-tuplet) by its converse.

$s_4{}^k$. Coupling to the left or to the right applied to a member of a pair $\mathfrak{p}$ ($\mathfrak{p}$ being a k-tuplet).

(Note that $s_3{}^k$ is the application of a process $s_1{}^h$ $(h<k)$ and $s_4{}^k$ the application of a process $s_2{}^h$ $(h<k)$ to one or other of the members of a pair which is a k-tuplet.)

7. If C is a class of k-tuplets such that a process $\mathbf{P}$, which is one of the steps $s_1{}^k$, $s_2{}^k$, $s_3{}^k$, $s_4{}^k$, can be applied to each of its elements, then the class exists whose elements are the k-tuplets arising from the elements of C by the process $\mathbf{P}$.

In order to prove 6 we proceed as follows. We first prove that, in the case of any k-tuplet, if $\mathfrak{a}$ is a member of degree higher than 1, we can, by means of the processes $s_1{}^k$ and $s_2{}^k$, lower the degree of $\mathfrak{a}$ by one. Indeed the k-tuplet, of which $\mathfrak{a}$ is a member of second or higher degree, must have one of the forms $\langle\mathfrak{p}, \langle\mathfrak{q}, \mathfrak{r}\rangle\rangle$ or $\langle\langle\mathfrak{q}, \mathfrak{r}\rangle, \mathfrak{p}\rangle$, $\mathfrak{a}$ being part of $\langle\mathfrak{q}, \mathfrak{r}\rangle$. If it has the form $\langle\mathfrak{p}, \langle\mathfrak{q}, \mathfrak{r}\rangle\rangle$ and $\mathfrak{a}$ is either part of $\mathfrak{r}$ or $\mathfrak{r}$ itself, we get by coupling to the left the k-tuplet $\langle\langle\mathfrak{p}, \mathfrak{q}\rangle, \mathfrak{r}\rangle$, in which the degree of $\mathfrak{a}$ is less by one—since in $\langle\langle \mathfrak{p}, \mathfrak{q}\rangle, \mathfrak{r}\rangle$ the number of brackets enclosing $\mathfrak{r}$, and therefore also the number of brackets enclosing $\mathfrak{a}$, is less by one than in $\langle \mathfrak{p}, \langle \mathfrak{q}, \mathfrak{r}\rangle\rangle$. If it has the form $\langle \mathfrak{p}, \langle \mathfrak{q}, \mathfrak{r}\rangle\rangle$ and $\mathfrak{a}$ is either part of $\mathfrak{q}$ or $\mathfrak{q}$ itself, we get, by taking the converse of the pair $\langle \mathfrak{p}, \langle \mathfrak{q}, \mathfrak{r}\rangle\rangle$ and then coupling to the right, the k-tuplet $\langle \mathfrak{q}, \langle \mathfrak{r}, \mathfrak{p}\rangle\rangle$, in which the degree of $\mathfrak{a}$ is less by one. And the case that the given k-tuplet has the form $\langle\langle \mathfrak{q}, \mathfrak{r}\rangle, \mathfrak{p}\rangle$ is handled in an entirely analogous way.

Being able, by means of the processes s_1^k, s_2^k to lower the degree of a member $\mathfrak{a}$ of a k-tuplet by one, as long as it is higher than 1, we can, by iterated applications of these processes, bring the degree down to 1. The k-tuplet we obtain in this way has one of the forms $\langle \mathfrak{a}, \mathfrak{b}\rangle$ or $\langle \mathfrak{b}, \mathfrak{a}\rangle$; and from $\langle \mathfrak{a}, \mathfrak{b}\rangle$ we can pass to $\langle \mathfrak{b}, \mathfrak{a}\rangle$ by the process s_1^k. Thus from a k-tuplet of which $\mathfrak{a}$ is a member $(k>1)$ we can pass by means of the processes s_1^k, s_2^k to a k-tuplet of the form $\langle \mathfrak{b}, \mathfrak{a}\rangle$.

Now in order to pass from a given k-tuplet to the normal k-tuplet,

$$\langle \mathfrak{a}_1, \langle \mathfrak{a}_2, \cdots, \langle \mathfrak{a}_{k-1}, \mathfrak{a}_k\rangle \cdots \rangle\rangle,$$

where $\mathfrak{a}_1, \cdots, \mathfrak{a}_k$ are the members of the given k-tuplet in some arbitrarily chosen order, we can proceed as follows.

First, by applying the processes s_1^k, s_2^k, we can pass from the given k-tuplet to a k-tuplet $\langle \mathfrak{b}, \mathfrak{a}_k\rangle$, where $\mathfrak{b}$ is a $(k-1)$-tuplet having $\mathfrak{a}_1, \cdots, \mathfrak{a}_{k-1}$ as its members. If $k=2$, then $\langle \mathfrak{b}, \mathfrak{a}_k\rangle$ is already the normal k-tuplet desired.

Let k be greater than 2. Then by applying the processes s_1^{k-1}, s_2^{k-1} to $\mathfrak{b}$ we can pass from $\mathfrak{b}$ to a $(k-1)$-tuplet $\langle \mathfrak{c}, \mathfrak{a}_{k-1}\rangle$, where $\mathfrak{c}$ is a $(k-2)$-tuplet having the members $\mathfrak{a}_1, \cdots, \mathfrak{a}_{k-2}$. This operation, carried out within the k-tuplet $\langle \mathfrak{b}, \mathfrak{a}_k\rangle$, leads to $\langle\langle \mathfrak{c}, \mathfrak{a}_{k-1}\rangle, \mathfrak{a}_k\rangle$. The processes involved are applications of s_1^{k-1} and s_2^{k-1} to one of the members of a pair (the pair being a k-tuplet); i.e. they are applications of s_3^k and s_4^k. Then from $\langle\langle \mathfrak{c}, \mathfrak{a}_{k-1}\rangle, \mathfrak{a}_k\rangle$ by coupling to the right we get $\langle \mathfrak{c}, \langle \mathfrak{a}_{k-1}, \mathfrak{a}_k\rangle\rangle$. If $k=3$, then this k-tuplet is already the one desired.

Let k be greater than 3. Then by the processes s_1^{k-2}, s_2^{k-2} applied to $\mathfrak{c}$ we can pass from $\mathfrak{c}$ to a $(k-2)$-tuplet $\langle \mathfrak{d}, \mathfrak{a}_{k-2}\rangle$, where $\mathfrak{d}$ is a $(k-3)$-tuplet having the members $\mathfrak{a}_1, \cdots, \mathfrak{a}_{k-3}$. Therefore by the processes s_3^k, s_4^k we can pass from $\langle \mathfrak{c}, \langle \mathfrak{a}_{k-1}, \mathfrak{a}_k\rangle\rangle$ to $\langle\langle \mathfrak{d}, \mathfrak{a}_{k-2}\rangle, \langle \mathfrak{a}_{k-1}, \mathfrak{a}_k\rangle\rangle$, and from this by coupling to the right we get $\langle \mathfrak{d}, \langle \mathfrak{a}_{k-2}, \langle \mathfrak{a}_{k-1}, \mathfrak{a}_k\rangle\rangle\rangle$. If $k=4$, then this is the normal k-tuplet desired.

If k is greater than 4 we continue in the same way.

After at most $k-1$ repetitions we will come to the desired normal k-tuplet,

$$\langle \mathfrak{a}_1, \langle \mathfrak{a}_2, \cdots, \langle \mathfrak{a}_{k-1}, \mathfrak{a}_k\rangle \cdots \rangle\rangle,$$

the steps applied all being of the kinds s_1^k, s_2^k, s_3^k, or s_4^k.

Turning now to the proof of 7, we note that if $\mathbf{P}$ is one of the processes s_1^k, s_2^k the desired result follows immediately from the axioms III c(2), c(3) and consequence 5 of these axioms (§2). Thus we need consider only the cases that $\mathbf{P}$ is s_3^k or s_4^k.

LEMMA. *If A and B are classes of pairs, there exists the class of all those pairs $\langle a, b\rangle$ for which there exists a set x such that $\langle a, x\rangle \eta A$ and $\langle x, b\rangle \eta B$.*

Indeed by the axiom III b(3) there exists the class C of all pairs $\langle\langle a, c\rangle, b\rangle$

such that $\langle a, c\rangle \eta A$. By the axioms III b(3), c(2) there exists the class of all pairs $\langle a, \langle c, b\rangle\rangle$ such that $\langle c, b\rangle \eta B$, and hence by III c(3) the class D of all pairs $\langle\langle a, c\rangle, b\rangle$ such that $\langle c, b\rangle \eta B$. The intersection of C and D is the class of all sets $\langle\langle a, c\rangle, b\rangle$ such that $\langle a, c\rangle \eta A$ and $\langle c, b\rangle \eta B$. Taking the converse class of this class and coupling to the left, we obtain the class H of those pairs $\langle\langle b, a\rangle, c\rangle$ for which $\langle a, c\rangle \eta A$ and $\langle c, b\rangle \eta B$. The converse class of the domain of H is the class whose existence is asserted by the lemma.

The operation of passing from a class A of pairs and a class B of pairs to the class of those pairs $\langle a, b\rangle$ for which there exists an x such that $\langle a, x\rangle \eta A$ and $\langle x, b\rangle \eta B$ (i.e. the class whose existence is asserted by the lemma just proved) will be called *composition of A with B*, and the lemma will be called the *composition lemma.*

The assertion 7, for the case that **P** is one of the processes $s_3{}^k$, $s_4{}^k$, is easily reduced, by means of the composition lemma and the axiom III c(2), to the two following assertions:

8. There exists the class of all sets of the form $\langle\langle a, b\rangle, \langle b, a\rangle\rangle$.

9. There exists the class of all sets of the form $\langle\langle a, \langle b, c\rangle\rangle, \langle\langle a, b\rangle, c\rangle\rangle$.

Proof of 8 is as follows. By consequence 6 of the axioms III, and the axioms III b(3), c(2), c(3), there exists the class of all sets of the form $\langle\langle\langle a, b\rangle, b\rangle, c\rangle$. From this class we obtain by coupling to the right the class of sets having the form $\langle\langle a, b\rangle, \langle b, c\rangle\rangle$. The intersection of this class with its converse is the class of all sets of the form $\langle\langle a, b\rangle, \langle b, a\rangle\rangle$.

Proof of 9 is as follows. By composition of the class of all sets of the form $\langle\langle a, b\rangle, \langle b, a\rangle\rangle$ with itself, we obtain the class of all sets of the form[12] $\langle\langle a, b\rangle, \langle a, b\rangle\rangle$. From this class we get by coupling to the left the class of all sets of the form $\langle\langle\langle a, b\rangle, a\rangle, b\rangle$, and hence, applying the axiom III b(3) and twice coupling to the right, the class of all sets of the form $\langle\langle a, b\rangle, \langle a, \langle b, c\rangle\rangle\rangle$. Taking the converse of the latter class, applying III b(3), and then again coupling to the right, we obtain the class M of all sets which have the form

$$\langle\langle a, \langle b, c\rangle\rangle, \langle\langle a, b\rangle, d\rangle\rangle.$$

On the other hand as we have seen (in the proof of 8) there exists the class of all sets of the form $\langle\langle b, c\rangle, \langle c, e\rangle\rangle$. From this class, applying III b(3), c(2), and coupling to the left, we obtain the class of all sets of the form $\langle\langle a, \langle b, c\rangle\rangle, \langle c, e\rangle\rangle$. And by composition of this class with the class of sets of the form $\langle\langle a, b\rangle, \langle b, a\rangle\rangle$ we obtain the class N of all sets of the form

$$\langle\langle a, \langle b, c\rangle\rangle, \langle e, c\rangle\rangle.$$

The intersection of M and N (III a(3)) is the class of all sets of the form

$$\langle\langle a, \langle b, c\rangle\rangle, \langle\langle a, b\rangle, c\rangle\rangle.$$

Herewith the demonstration of our theorem on the correspondence between predicates and classes is completed. We shall call this theorem briefly the *class*

[12] Of course this class can be shown to exist also in other ways. For example it can be obtained as the intersection of the class of all pairs $\langle c, c\rangle$ with the class of those pairs $\langle r, s\rangle$ in which r belongs to the class of all pairs.

theorem; in the further development of the present system we shall have occasion to use it constantly.

The following immediate consequences may be stated at once.

Corresponding to any class C there exists:

1) A class which is the intersection of the elements of C, i.e., a class whose elements are the sets which are in every element of C.

2) A class which is the sum of the elements of C, i.e., a class whose elements are the sets which are in at least one element of C.

3) A class whose elements are the subsets of C.

The constitutive expressions with argument a and parameter C to which these classes correspond (under the class theorem) are, respectively,

$$(x)(x\eta C \rightarrow a\epsilon x),$$

$$(Ex)(x\eta C \;\&\; a\epsilon x),$$

$$(x)(x\epsilon a \rightarrow x\epsilon C).$$

Instead of the parameter C, referring to classes, we may take a parameter b, referring to sets, at the same time replacing η by ϵ; i.e., we may employ the constitutive expressions,

$$(x)(x\epsilon b \rightarrow a\epsilon x),$$

$$(Ex)(x\epsilon b \;\&\; a\epsilon x),$$

$$(x)(x\epsilon a \rightarrow x\epsilon b).$$

Applying the class theorem to these, we find that corresponding to any set b there exists:

1) A class which is the intersection of the elements of b.

2) A class which is the sum of the elements of b.

3) The class of all subsets of b.

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